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Some Approximate Solutions to the Discrete Master Equation

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Recent mathematical developments on approximate diffusionlike solutions to the master equation are summarized. The technique is applied to two master equations of physical interest—one that describes the phenomenon of superradiance and a second that characterizes generation–recombination noise in semiconductors. For this second case, some previously obtained equilibrium results are found and the extension of these results to finite times is given.

KEY WORDS: Stochastic processes; master equation; diffusion approximation.

There are few general mathematical techniques known for the solution of large classes of master equations, the best known of these being the method of generating functions and the method of spectral decomposition for temporally homogeneous processes.[1] Other approximate techniques have been applied to problems in neutron thermalization and are discussed in some detail in the monograph by Williams.[2] Recently, some attention has been given by mathematicians to asymptotic solutions to master equations that arise from stochastic models that are characterized by large population sizes[3–10]. It is the purpose of this paper to summarize some results of the mathematical analysis and in particular to apply it to two master equations that have appeared in the literature of physics. Although the results that are obtained for these particular examples are themselves of interest, the more

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general applicability of the technique in physical problems is of greater interest.

The first of the equations to be discussed is the master equation describing the phenomenon of superradiance, that is, the radiation of photons from a collection of two-level atoms first analyzed by Dicke[11] and more recently discussed by a number of authors.[12–15] Let N be the total number of available atoms, let $p_n(t)$ be the probability that there are n atoms in an excited state, and let I be the radiation rate from a single atom in the cavity. Then, under assumptions carefully enumerated by Bonifacio *et al.*,[14] the equations satisfied by $p_n(t)$ read

$$\dot{p}_n(t) = g_{n-1} p_{n+1} - g_n p_n , \qquad n = 1, 2, \ldots, N \tag{1}$$

where $g_n = I_n(N + 1 - n)$ and N is the total number of atoms in the system. We shall be interested in the solution to this set of equations in the limit of large N. Although a solution to Eq. (1) in terms of Laplace transforms is possible, the results are quite complicated and not very easy to use. It is of some incidental interest that Eq. (1) describes what is known as the simple epidemic and has been in the literature of biometrics for approximately twenty years.[16–19]

In order to proceed with the analysis of Eq. (1), we note that a naive first approximation would be the so-called deterministic approximation in which one assumes that the number of excited atoms $m(t)$ decays according to[2]

$$\dot{m}(t) = -Im(t)[N - m(t)] \tag{2}$$

This equation has the solution

$$m(\tau) = Nm(0)\, e^{-\tau}/[N - m(0)(1 - e^{-\tau})] \tag{3}$$

in which $\tau = INt$. Since this approximation is only reasonable when $m(\tau)$ is large, we define

$$\rho(\tau) = m(\tau)/N = \rho_0 e^{-\tau}/[1 - \rho_0(1 - e^{-\tau})] \tag{4}$$

where $\rho(0) = \rho_0$. When $\rho_0 = 1$, that is, when all atoms are initially excited, one must use the more accurate expression for $\rho(\tau)$

$$\rho(\tau) = (N + 1)/(N + e^{\tau}) \approx N/(N + e^{\tau}) \tag{5}$$

as Eq. (4) reduces to a constant for $\rho_0 = 1$. Since $\rho(0) \leqslant 1$, the function $\rho(\tau)$ remains bounded by 1 for all time. The approximation of this paper, sug-

[2] More properly, Eq. (2) should read $\dot{m} = -\mathrm{Im}(t)[N + 1 - m(t)]$. This detail is only important when $m(0) \sim N$.

gested by the formulation given by McNeil,[9] is based on the following representation of $m(\tau)$, now assumed to be a random variable:

$$m(\tau) = N\rho(\tau) + Z\sqrt{N} \tag{6}$$

where the first term on the right is the deterministic result and the second is the fluctuating term assumed to be proportional to $N^{1/2}$ [i.e., Z is assumed to be $O(1)$]. We next derive an equation for the characteristic function

$$\begin{aligned}\psi(\theta, \tau) &= E \exp[i\theta z(\tau)] \\ &= \{\exp[-i\theta\rho(\tau)\sqrt{N}]\} E \exp[i\theta m(\tau)/\sqrt{N}]\end{aligned} \tag{7}$$

representing the parameter I as α/N, where α is $O(1)$. This scaling of I can be motivated by referring to Eq. (3), in which time occurs in the dimensionless combination NIt.

The function $E \exp[i\theta M(\tau)/\sqrt{N}]$ is just the generating function

$$A(\theta, \tau) = \sum_{n=0}^{\infty} p_n \exp(i\theta n/\sqrt{N}) \tag{8}$$

by definition, and satisfies

$$\frac{\partial A}{\partial \tau} = \frac{1}{N}\left[\exp\left(-\frac{i\theta}{N^{1/2}}\right) - 1\right]\left[-i(N^{1/2} + N^{3/2})\frac{\partial A}{\partial \theta} + N\frac{\partial^2 A}{\partial \theta^2}\right] \tag{9}$$

Hence, by Eq. (7), the function $\psi(\theta, \tau)$ is the solution of

$$\begin{aligned}&\frac{\partial \psi}{\partial \tau} + i\theta N^{1/2}\dot{\rho}\psi \\ &\quad = \frac{1}{N}\left[\exp\left(-\frac{i\theta}{N^{1/2}}\right) - 1\right]\left[-i(N^{1/2} + N^{3/2})\left(\frac{\partial \psi}{\partial \theta} + i\rho N^{1/2}\psi\right)\right. \\ &\qquad \left. + N\left(\frac{\partial^2 \psi}{\partial \theta^2} + 2i\rho N^{1/2}\frac{\partial \psi}{\partial \theta} - N\rho^2\psi\right)\right]\end{aligned} \tag{10}$$

If we now expand the term $\exp(-i\theta/\sqrt{N})$ into a Taylor series in θ and collect the coefficients of various powers of N, we find

$$O(N^{1/2}) \text{ term:} \qquad i\theta\psi[\dot{\rho} + \rho(1 - \rho)] = 0 \tag{11a}$$

$$O(1) \text{ term:} \qquad \frac{\partial \psi}{\partial \tau} + \theta[1 - 2\rho(\tau)]\frac{\partial \psi}{\partial \theta} = -\frac{\theta^2}{2}\rho(\tau)[1 - \rho(\tau)]\psi \tag{11b}$$

Further terms are $O(N^{-1/2})$ or smaller. Equation (11a) is just seen to be equivalent to the deterministic equation. To solve Eq. (11b), we substitute a trial solution of the form

$$\psi(\theta, \tau) = \exp[-\theta^2\Gamma(\tau)/2] \tag{12}$$

where $\Gamma(\tau)$ is to be determined. The function $\Gamma(\tau)$ is found to satisfy

$$\dot{\Gamma}(\tau) + 2[1 - 2\rho(\tau)]\,\Gamma(\tau) = \rho(\tau)[1 - \rho(\tau)] \tag{13}$$

subject to $\Gamma(0) = \Gamma_0$. The solution to this equation is found to be

$$\Gamma(\tau) = \frac{\Gamma_0 e^{-2\tau}}{(1 - \rho_0 + \rho_0 e^{-\tau})^4} + \frac{\rho_0(1 - \rho_0)\, e^{-2\tau}}{(1 - \rho_0 + \rho_0 e^{-\tau})^4} \times [(1 - \rho_0)^2 (e^{\tau} - 1) + 2\rho_0(1 - \rho_0)\,\tau + \rho_0^2(1 - e^{-\tau})] \tag{14}$$

for $\rho_0 < 1$ and

$$\Gamma(\tau) = [Ne^{-2\tau}/(Ne^{-\tau} + 1)^4][\tfrac{1}{2}N^2(1 - e^{-\tau}) + 2N(\tau - 1 + e^{-\tau}) + e^{\tau} - 1 - \tau] \tag{15}$$

for $\rho_0 = 1$, where we have neglected terms that are $O(1/N)$. It follows from the representation of Eq. (7) that p_n is given by

$$\begin{aligned} p_n &\sim (1/2\pi) \int_{-\pi}^{\pi} \exp[-N\varphi^2\Gamma(\tau)/2] \exp[i(N\rho - n)\,\varphi]\, d\varphi \\ &\sim (1/2\pi) \int_{-\infty}^{\infty} \exp[-N\varphi^2\Gamma(\tau)/2] \cos[N\rho(\tau) - n]\,\varphi\, d\varphi \\ &= [1/(2\pi N)^{1/2}\,\Gamma(\tau)] \exp\{-[n - N\rho(\tau)]^2/(2N\Gamma(\tau))\} \end{aligned} \tag{16}$$

that is to say, the probability distribution is Gaussian, peaked at the deterministic mean $N\rho(\tau)$, with a variance given by

$$\sigma^2 = N\Gamma(\tau) \tag{17}$$

This function shows an initial broadening and a final sharpening of the peak as more and more atoms leave the excited state. The normalized mean intensity $\langle I(t)\rangle/I$ can be found directly in the present approximation from the expression for $\psi(\theta, \tau)$. The definition of $\langle I(t)\rangle/I$ is

$$\begin{aligned} \langle I(t)\rangle/I &= \sum_n g_n p_n(t) = \sum_n n(N + 1 - n)\, p_n(t) \\ &= -i(N^{1/2} + N^{3/2}) \frac{d}{d\theta} [\psi \exp(i\rho\theta\sqrt{N})]\Big|_{\theta=0+} \\ &\quad + N \frac{d^2}{d\theta^2} [\psi \exp(i\rho\theta\sqrt{N})]\Big|_{\theta=0+} \\ &\sim N^2\rho(\tau)[1 - \rho(\tau)] - N[\rho(\tau) - \Gamma(\tau)] \end{aligned} \tag{18}$$

Since $I = \alpha/N$, we have for $\langle I(\tau)\rangle$.

$$\langle I(\tau)\rangle = [\alpha N\rho_0(1-\rho_0)\,e^{-\tau}/(1-\rho_0+\rho_0 e^{-\tau})^2] - O(1)$$
$$= I(0)\,e^{\tau}/(\rho_0^2\{1+[I(0)/\alpha N\rho_0^2]\,e^{\tau}\}^2) \tag{19}$$

where $I(0)$ is the initial intensity.

In the limit $\rho_0 = 1$, this reduces to the expression derived by Degiorgio and Ghielmetti.[15] Notice that the first term on the right-hand side can be derived from the deterministic theory, while the contribution from the more accurate stochastic theory is of lower order in N. In the same way that we have derived an expression for the relative mean intensity, we can calculate the relative variance

$$\Delta(\tau) = \frac{1}{(\langle I(\tau)\rangle)^2}\sum_n g_{n-1}g_n p_n(\tau) - 1$$
$$= \frac{1}{N}\left[\Gamma(6\rho^2-6\rho+1) - \rho(1-\rho)(1-4\rho) + \frac{2(\rho-\Gamma)}{1-\rho}\right] + O\left(\frac{1}{N^2}\right) \tag{20}$$

which goes to zero in the limit of large N. Thus, as is the case in the stochastic theory of chemical reactions,[20,21] there are no macroscopic fluctuations that can be attributed to the stochastic nature of the system for N sufficiently large.

It is of peripheral interest that in the present diffusion approximation, the expected number of excited atoms is $\langle n\rangle = N\rho$, i.e., it is equal to the deterministic result. A more accurate calculation leads to correction terms in powers of N^{-1}, and in fact, one can derive an asymptotic expansion for the moments of the form

$$\mu_n = \mu_n^{(0)} + \frac{\mu_n^{(1)}}{N} + \frac{\mu_n^{(2)}}{N^2} + \cdots \tag{21}$$

in which the $\mu_n^{(0)}$ corresponds to results of a deterministic theory and the $\mu_n^{(j)}$ are higher-order corrections that can be calculated recursively. Because these corrections are not important in the context of the present phenomena, details of the calculation are deferred to the appendix.

The second example to be analyzed in the present framework is the master equation describing generation–recombination noise in two-level semiconductors. The relevant equations were derived by Burgess[22,23] and a discussion of them is to be found in the review article by van Vliet and Fassett.[24] In this system, there are two possible energy levels for a fixed number N of electrons. Let $n(t)$ electrons be the (random) number of electrons

in one of the energy levels at time t and let $p_r(t) = \Pr\{n(t) = r\}$. The master equation describing the dynamics of this type of noise has the general form

$$p_n = g(n-1)\,p_{n-1} + r(n+1)\,p_{n+1} - [\,g(n) + r(n)]\,p_n \tag{22}$$

where $g(n)$ and $r(n)$ are time-independent rate constants, the generation and recombination rates, respectively. Typical functional dependences of these rates are

$$r(n) = An^s, \qquad g(n) = B(N-n)^s \tag{23}$$

where A, B, and s are constants. For the problems discussed in the literature, the parameter s has either been 1 or 2. Equation (22) differs from Eq. (1) in that transitions are allowed in both directions in the noise problems. The quantities of principal interest are $n(\infty)$, $\sigma^2(\infty) = \langle[n(\infty) - \langle n(\infty)\rangle]^2\rangle$, and the time-dependent autocorrelation function for the random variable $n(t) - n(\infty)$. Heuristic approximations to these quantities have been derived by Burgess and Van Vliet and Fassett that are valid in the limit of large N. The results assume that $r(j)$ and $g(j)$ are analytic functions of j and are summarized as follows: $\langle n(\infty)\rangle$ is found as the root of

$$g(\langle n\rangle) = r(\langle n\rangle) \tag{24}$$

σ^2 is given by

$$\sigma^2 = r(\langle n\rangle)/[r'(\langle n\rangle) - g'(\langle n\rangle)] \tag{25}$$

and the autocorrelation function of the random variable $n(\tau) - n(\infty)$ is approximated by

$$C(\tau) = \sigma^2 e^{-t/T}, \qquad T = [r'(\langle n\rangle) - g'(\langle n\rangle)]^{-1}. \tag{26}$$

To my knowledge, these approximate results have not been checked by other means.

Rather than repeat the analysis corresponding to Eqs. (10) and (11), I will merely quote the results given by Stone[25] and Iglehart.[4] These are that, provided $g(n)$ and $r(b)$ have the asymptotic (in N) forms

$$\begin{aligned} g(aN + x\sqrt{N}) &= \alpha(a)\,N + \beta_1(a)\,x\sqrt{N} + O(1) \\ r(aN + x\sqrt{N}) &= \alpha(a)\,N + \beta_2(a)\,x\sqrt{N} + O(1) \end{aligned} \tag{27}$$

the distribution of the random variable

$$Z = [n(t) - aN]/\sqrt{N} \tag{28}$$

converges as $N \to \infty$ to an Ornstein–Uhlenbeck process whose probability density ρ is the solution to

$$\partial\rho/\partial t = \alpha(a)(\partial^2\rho/\partial x^2) + [\beta_2(a) - \beta_1(a)](\partial/\partial x)(x\rho) \tag{29}$$

The solution to this equation is a Gaussian function with a mean and variance given by

$$\begin{aligned} E\{z(t)\} &= z(0)\, e^{-(\beta_2-\beta_1)t} \\ \operatorname{var}\{z(t)\} &= [\alpha/(\beta_2-\beta_1)][1-e^{-2(\beta_2-\beta_1)t}] \end{aligned} \tag{30}$$

A heuristic proof of these results can be given that parallels that of the derivation of Eq. (11). In order to see how the results can be applied in a specific case, let us consider the forms for generation and recombination rates shown in Eq. (23).

It is straightforward to calculate from Eq. (23) that

$$\begin{aligned} r(aN+x\sqrt{N}) &= AN^s[a^s+(sxa^{s-1}/\sqrt{N})+\cdots] \\ g(aN+x\sqrt{N}) &= BN^s\{(1-a)^s-[sx(1-a)^{s-1}/\sqrt{N}]+\cdots\} \end{aligned} \tag{31}$$

Hence, in order for Eq. (27) to be satisfied, we must assume that A and B are scaled like $A = A_0N^{1-s}$ and $B = B_0N^{1-s}$, in which A_0 and B_0 are $O(1)$. This scaling procedure is also suggested by an examination of the deterministic kinetic equations. Since the coefficients of N in Eq. (27) are equal, the constant a is determined from $A_0^{1/s}\,a = B_0^{1/s}(1-a)$, or

$$a = B_0^{1/s}/(A_0^{1/s}+B_0^{1/s}) \tag{32}$$

so that

$$\begin{aligned} \alpha(a) &= A_0B_0/(A_0^{1/s}+B_0^{1/s})^s \\ \beta_1(a) &= -sB_0(1-a)^{s-1}, \qquad \beta_2(a) = sA_0a^{s-1} \end{aligned} \tag{33}$$

With these results, we can calculate for the mean and variance of $n(t)$

$$\begin{aligned} \langle n(t)\rangle &= aN+[n(0)-aN]\exp[-sA_0a^{s-1}t/(1-a)] \\ \sigma^2(t) &= [Na(1-a)/s]\{1-\exp[-2a^{s-1}A_0st/(1-a)]\} \end{aligned} \tag{34}$$

Furthermore, the autocorrelation function calculated from Eq. (34) agrees with the result given by van Vliet and Fassett[21] and obtained by other means. The results for $\langle n(t)\rangle$ and $\sigma^2(t)$ given in Eq. (34) agree with the approximations derived by Burgess for $t = \infty$ and extend his results to finite values of time.

It is evident from the examples that we have just discussed that the diffusion approximation can be used to solve some quite general master equations provided that the number of levels tend to infinity and provided that the rates are properly scaled. For example, the method just discussed is readily extended to the case of noise in three-level semiconductors for which equilibrium variances have been calculated.[26] The theory so derived would

allow one to develop a full time-dependent theory rather than being restricted to the asymptotic regime.

The method outlined here is related to that of van Kampen,[27] truncated at his first approximation. It is not clear that his higher approximations are correct because they do not appear to allow the analog of Eq. (21). However, it may also be the case that most physical problems have such large numbers involved that little more than a deterministic approach[20,21] is required for the analysis. This is no longer true for genetic or ecological applications, where population sizes are order of magnitudes lower than atomic numbers.

APPENDIX: EXPANSION OF THE MOMENTS OF EQ. (1) IN POWERS OF N^{-1}

In this appendix, we discuss higher approximations in the calculation of moments of the supperradiance equation. That is to say, the expectation $\langle n \rangle$ calculated from Eq. (16) is $\langle n \rangle = N\rho$, which is just the result expected from a deterministic equation for n. When small numbers of atoms are involved, we expect an expansion of the form

$$\langle n \rangle = N\rho + \rho_1 + (\rho_2/N) + (\rho_3/N^2) + \cdots .$$

Bailey[19] has given an analogous calculation for the simple epidemic; the present calculation represents a considerable simplification over Bailey's technique.

The analysis starts by considering the generating function

$$U(\theta, \tau) = \sum_{n=0}^{\infty} p_n e^{-n\theta/N} \tag{A.1}$$

where $\tau = NIt$, as in the text. This function satisfies the equation

$$\begin{aligned} \frac{\partial U}{\partial \tau} &= -N(e^{\theta/N} - 1)\left(\frac{\partial U}{\partial \theta} + \frac{\partial^2 U}{\partial \theta^2}\right) \\ &= -\left(\theta + \frac{\theta^2}{2N} + \frac{\theta^3}{6N^2} + \cdots\right)\left(\frac{\partial U}{\partial \theta} + \frac{\partial^2 U}{\partial \theta^2}\right) \end{aligned} \tag{A.2}$$

If we now assume an expansion for U in the form

$$U = U_0 + (U_1/N) + (U_2/N^2) + \cdots \tag{A.3}$$

and equate successive powers of N^{-1}, we find that

$$\frac{\partial U_0}{\partial \tau} = -\theta\left(\frac{\partial U_0}{\partial \theta} + \frac{\partial^2 U_0}{\partial \theta^2}\right) \tag{A.4a}$$

$$\frac{\partial U_1}{\partial \tau} + \theta\left(\frac{\partial U_1}{\partial \theta} + \frac{\partial^2 U_1}{\partial \theta^2}\right) = -\frac{\theta^2}{2}\left(\frac{\partial U_0}{\partial \theta} + \frac{\partial^2 U_0}{\partial \theta^2}\right) \tag{A.4b}$$

If we assume that

$$U_0(\theta, 0) = U(\theta, 0) = e^{-n_0\theta/N}, \qquad U_1(\theta, 0) = U_2(\theta, 0) = \cdots = 0 \tag{A.5}$$

then the solution to Eq. (A.4a) is

$$U_0(\theta, \tau) = e^{-\theta\rho(\tau)} \tag{A.6}$$

where $\rho(\tau)$ is given in Eq. (4) of the text. Thus, $U_0(\theta, \tau)$ gives the result of the deterministic, or kinetic, theory. Higher-order terms in the sequence of Eq. (A.4) are all of the form

$$\frac{\partial U}{\partial \tau} + \theta\left(\frac{\partial U}{\partial \theta} - \frac{\partial^2 U}{\partial \theta^2}\right) = F(\theta, \tau) \tag{A.7}$$

where, at any order, the function $F(\theta, \tau)$ is known from lower-order terms. Thus, for example,

$$F_1(\theta, \tau) = -\frac{\theta^2}{2}\left(\frac{\partial U_0}{\partial \theta} + \frac{\partial^2 U_0}{\partial \theta^2}\right) = -\frac{\theta^2}{2}\dot{\rho}(\tau)\, e^{-\theta\rho(\tau)} \tag{A.8}$$

Although it is possible to derive a solution by Laplace transforms, the structure of Eq. (A.8) suggests that we substitute a solution of the form

$$U_1(\theta, \tau) = A(\tau)\,\theta e^{-\theta\rho(\tau)} + B(\tau)\,\theta^2 e^{-\theta\rho(\tau)} \tag{A.9}$$

It is easy to verify that $A(\tau)$ and $B(\tau)$ must be the solutions to

$$\begin{aligned} \dot{B} + 2[1 - 2\rho(\tau)]\,B &= \rho(\tau)[1 - \rho(\tau)]/2 \\ \dot{A} + [1 - 2\rho(\tau)]\,A &= -2B \end{aligned} \tag{A.10}$$

subject to $A(0) = B(0) = 0$. The solutions to these equations are

$$\begin{aligned} B(\tau) = \frac{1}{2}\left\{\exp\left[-2\tau + 4\int_0^\tau \rho(v)\,dv\right]\right\} \int_0^\tau \rho(v)[1 - \rho(v)] \\ \times \exp\left[2v - 4\int_0^v \rho(v')\,dv'\right] dv \end{aligned} \tag{A.11}$$

$$A(\tau) = -2\left\{\exp\left[-\tau + 2\int_0^\tau \rho(v)\,dv\right]\right\} \int_0^\tau B(v) \exp\left[v - 2\int_0^v \rho(v')\,dv'\right] dv$$

The combination of Eqs. (A.9) and (A.11) constitutes a generalization of Bailey's results and allows the proper calculation of N^{-1} corrections to the moments.

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